Foldings in studying flag manifolds

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Motivating triangle

Combinatorics

$\Phi(X_P) = P \subset \mathbb{R}^N$
moment polytope

moment map
$\Phi \colon X_P \to \mathbb{R}^N$

$|P \cap \mathbb{Z}^N|$
$= \dim_{\mathbb{C}} H^0(X_P, \mathcal{L})$

Topology/Geometry

$(X_P, \mathcal{L}) \hookrightarrow \mathbb{P}^m$
smooth projective toric variety
of $\mathbb{C}$-dim N

taking sections
$\longleftrightarrow$

Representation Theory

$(S^1)^N \circlearrowright H^0(X_P, \mathcal{L})$

Question

Can we build an analogous picture for general $X^N \hookrightarrow \mathbb{P}^m$?

Motivating triangle

Combinatorics

$\Delta_{\boldsymbol{i}}(\lambda) \subset \mathbb{R}^N$
string polytopes

Neweton–Okounkov
body of
$(G/B, \mathcal{L}_\lambda, \nu_{\boldsymbol{i}})$

$|\Delta_{\boldsymbol{i}}(\lambda) \cap \mathbb{Z}^N|$
$= \dim_{\mathbb{C}} V_\lambda^*$

Borel–Weil–Bott theorem

Topology/Geometry $\longleftrightarrow$ Representation Theory

$(G/B, \mathcal{L}_\lambda)$

$G \circlearrowright H^0(G/B, \mathcal{L}_\lambda) \cong V_\lambda^*$

Question

Can we build an analogous picture for general $X^N \hookrightarrow \mathbb{P}^m$?
Which topological/geometric data can be obtained from $\Delta_{\boldsymbol{i}}(\lambda)$?

What are string polytopes?

- G: simply-connected semisimple algebraic group over $\mathbb{C}$ of type X_n. Today, X is A, B, or C.
- $\boldsymbol{i}$: reduced decomposition of the longest element of the Weyl group of G.
- λ: dominant integral weight.

Using these data, the string polytope $\Delta_{\boldsymbol{i}}(\lambda)$ is defined in [Littelmann, 98], which

1. is a rational polytope lives in $\mathbb{R}^N$, where $N = \dim_{\mathbb{C}} G/B$ (if G is of type A, then $N = \frac{n(n+1)}{2}$; if G is of type B or C, then $N = n^2$),
2. $\Delta_{\boldsymbol{i}}(\lambda) \cap \mathbb{Z}^N \leftrightarrow$ weights of $V(\lambda)$,
3. is a Newton–Okounkov body of $(G/B, \mathcal{L}_\lambda, \nu_{\boldsymbol{i}})$ (by [Kaveh, 15]).
4. For $\boldsymbol{i}_A = (1, 2, 1, 3, 2, 1, ..., n, n-1, \dots, 1)$ in type A_n,

$$\Delta_{\boldsymbol{i}}(\rho) \simeq \text{Gelfand–Tsetlin polytope } \mathrm{GT}(\rho).$$

For $\boldsymbol{i}_C = (n, n-1, n, n-1, n-2, n-1, n, n-1, n-2, \dots, 1, 2, \dots, n, \dots, 2, 1)$ in type C_n,

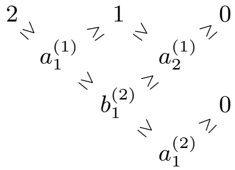
$$\Delta_{\boldsymbol{i}}(\rho) \simeq \text{Gelfand–Tsetlin polytope } \mathrm{GT}_C(\rho).$$

Here, ρ is the sum of fundamental weights in each case (by [Littelmann, 98]).

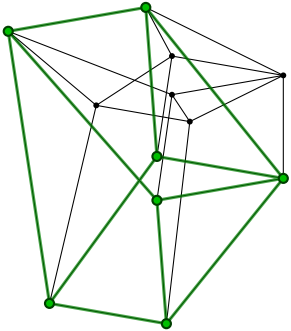
Combinatorics of $\Delta_{\boldsymbol{i}}(\lambda)$ depends on $\boldsymbol{i}$.

Gelfand–Tsetlin polytopes

$G = \mathsf{Sp}_4(\mathbb{C}), \lambda = \rho = \varpi_1 + \varpi_2.$



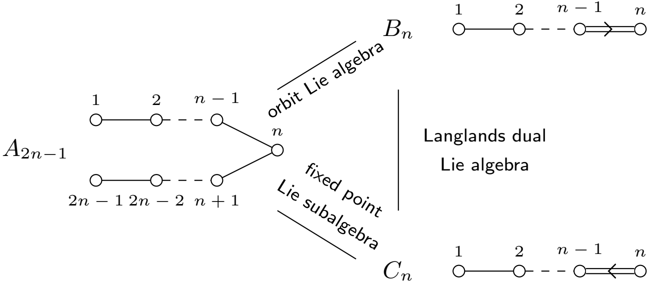
The f-vector of $\mathsf{GT}_C(\rho)$ is $(1, 12, 26, 22, 8, 1)$.



$a_1^{(2)} = 0$ defines a 3-dimensional Gelfand–Testlin polytope of type A.

Description of string polytopes

When G is of type A, [Gleizer–Postnikov, 00] provided a description of $\Delta_{\boldsymbol{i}}(\lambda)$ using a *wiring diagram* given by $\boldsymbol{i}$. However, such combinatorial descriptions are not known yet for other Lie types. On the other hand, the Lie algebras of type A, B, and C have the following relation.



Using the above relation, [Fujita, 18] studied a *folding* procedure for string cones.

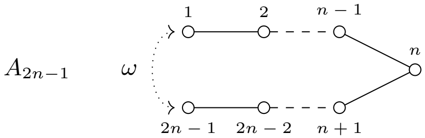
GOAL

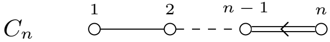
1. Providing an explicit description of string polytopes in types B and C.
2. Characterizing *Gelfand–Tsetlin type* string polytopes in type C.

Fixed point Lie subalgebras

$$\Delta_{\boldsymbol{i}}(\lambda) = \mathcal{C}_{\boldsymbol{i}}^{(C_n)} \cap \mathcal{C}_{\boldsymbol{i}}^{\lambda}$$

$\mathcal{C}_{\boldsymbol{i}}^{(C_n)}$ is called the string cone, $\mathcal{C}_{\boldsymbol{i}}^{\lambda}$ is called the λ-cone.





$$\omega(i) = 2n - i =: \bar{i}$$

Define a Lie algebra automorphism $\hat{\omega} \colon \mathfrak{sl}_{2n} \to \mathfrak{sl}_{2n}$ by $\hat{\omega}(X) = (\overline{w}_0)^{-1} \cdot (-X^T) \cdot \overline{w}_0$ for $X \in \mathfrak{sl}_{2n}$, where

$$\overline{w}_0 := \begin{pmatrix} 0 & 0 & 0 & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix} \in \mathfrak{sl}_{2n}.$$

Then $(\mathfrak{sl}_{2n})^{\hat{\omega}} := \{X \in \mathfrak{sl}_{2n} \mid \hat{w}(X) = X\} = \mathfrak{sp}_{2n}$.

Foldings in string cones

$\Theta\colon W^{(C_n)} \hookrightarrow \mathfrak{S}_{2n} = W^{(A_{2n-1})}$ given by $\Theta(s_i) = s_i s_{\bar{i}}$ if $i \neq n$; $\Theta(s_n) = s_n$. For instance, $\Theta(2,1,2,1) = (2,1,3,2,1,3)$. Define $\Omega_{\boldsymbol{i}}^{A,C} : \mathbb{R}^6 \to \mathbb{R}^4$ by

$$\Omega_{\boldsymbol{i}}^{A,C}(a_1, a_2, \bar{a}_2, a_3, a_4, \bar{a}_4) := (a_1, a_2 + \bar{a}_2, a_3, a_4 + \bar{a}_4).$$

On the other hand, $W^{(B_n)} = W^{(C_n)}$. Define $\Gamma_{\boldsymbol{i}}^{C,B} : \mathbb{R}^4 \to \mathbb{R}^4$ by

$$\Gamma_{\boldsymbol{i}}^{C,B}(a_1, a_2, a_3, a_4) := (2a_1, a_2, 2a_3, a_4).$$

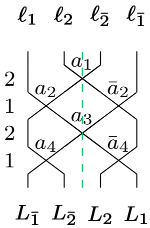
Theorem [Fujita, 18]

1. $\Omega_{\boldsymbol{i}}^{A,C}(\mathcal{C}_{\Theta(\boldsymbol{i})}^{(A_{2n-1})}) = \mathcal{C}_{\boldsymbol{i}}^{(C_n)}$.
2. $\Gamma_{\boldsymbol{i}}^{C,B}(\mathcal{C}_{\boldsymbol{i}}^{(C_n)}) = \mathcal{C}_{\boldsymbol{i}}^{(B_n)}$.

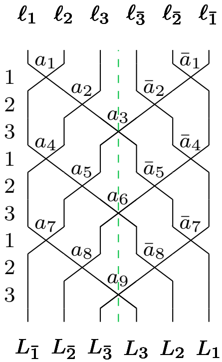
Accordingly, $\mathcal{C}_{\boldsymbol{i}}^{(C_n)}$ and $\mathcal{C}_{\boldsymbol{i}}^{(B_n)}$ are combinatorially same.

Symplectic wiring diagrams

$\boldsymbol{i} = (2,1,2,1)$ $\quad$ $\boldsymbol{i} = (1,2,3,1,2,3,1,2,3)$



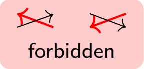
$G^{\mathrm{symp}}(2,1,2,1)$

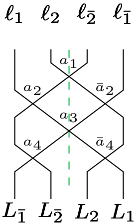


$G^{\mathrm{symp}}(1,2,3,1,2,3,1,2,3)$

Rigorous paths in symplectic wiring diagram

$\boldsymbol{i} = (2, 1, 2, 1)$





- $(\ell_1 \to \ell_2) \rightsquigarrow \bar{a}_4 \geq 0$
- $(\ell_2 \to \ell_{\bar{2}}) \rightsquigarrow a_1 \geq 0$
- $(\ell_2 \to \ell_1 \to \ell_{\bar{1}} \to \ell_{\bar{2}}) \rightsquigarrow a_3 - (a_4 + \bar{a}_4) \geq 0$
- $(\ell_2 \to \ell_{\bar{1}} \to \ell_1 \to \ell_{\bar{2}}) \rightsquigarrow (a_2 + \bar{a}_2) - a_3 \geq 0$
- $(\ell_2 \to \ell_1 \to \ell_{\bar{2}}) \rightsquigarrow \bar{a}_2 - a_4 \geq 0$
- $(\ell_2 \to \ell_{\bar{1}} \to \ell_{\bar{2}}) \rightsquigarrow a_2 - \bar{a}_4 \geq 0$
- $(\ell_{\bar{2}} \to \ell_{\bar{1}}) \rightsquigarrow a_4 \geq 0$

For a rigorous path P, define

$$\sum_{j=1}^{N} c_j a_j \geq 0, \quad \text{where } c_j = \begin{cases} 1 & \text{if } P \text{ travels from } \ell_r \to \ell_s \text{ at } a_j \text{ and } r < s, \\ -1 & \text{if } P \text{ travels from } \ell_r \to \ell_s \text{ at } a_j \text{ and } r > s, \\ 0 & \text{otherwise.} \end{cases}$$

String cone inequalities

$$\left.\begin{array}{l}
(\ell_1 \to \ell_2) \rightsquigarrow \bar{a}_4 \geq 0 \\
(\ell_2 \to \ell_{\bar{2}}) \rightsquigarrow a_1 \geq 0 \\
(\ell_2 \to \ell_1 \to \ell_{\bar{1}} \to \ell_{\bar{2}}) \rightsquigarrow a_3 - (a_4 + \bar{a}_4) \geq 0 \\
(\ell_2 \to \ell_{\bar{1}} \to \ell_1 \to \ell_{\bar{2}}) \rightsquigarrow (a_2 + \bar{a}_2) - a_3 \geq 0 \\
(\ell_2 \to \ell_1 \to \ell_{\bar{2}}) \rightsquigarrow \bar{a}_2 - a_4 \geq 0 \\
(\ell_2 \to \ell_{\bar{1}} \to \ell_{\bar{2}}) \rightsquigarrow a_2 - \bar{a}_4 \geq 0 \\
(\ell_{\bar{2}} \to \ell_{\bar{1}}) \rightsquigarrow a_4 \geq 0
\end{array}\right\} \text{define } \mathcal{C}_{\boldsymbol{i}}^{(A_3)} \subset \mathbb{R}^6$$

Using the projection map

$$\Omega_{\boldsymbol{i}}^{A,C}(a_1, a_2, \bar{a}_2, a_3, a_4, \bar{a}_4) = (a_1, a_2 + \bar{a}_2, a_3, a_4 + \bar{a}_4) =: (\mathsf{a}_1, \mathsf{a}_2, \mathsf{a}_3, \mathsf{a}_4) \in \mathbb{R}^4,$$

$$\mathcal{C}_{\boldsymbol{i}}^{(C_2)} = \Omega_{\boldsymbol{i}}^{A,C}(\mathcal{C}_{\Theta(\boldsymbol{i})}^{(A_3)}) = \{(\mathsf{a}_1, \mathsf{a}_2, \mathsf{a}_3, \mathsf{a}_4) \in \mathbb{R}^4 \mid \mathsf{a}_4 \geq 0, \mathsf{a}_1 \geq 0, \mathsf{a}_3 - \mathsf{a}_4 \geq 0, \mathsf{a}_2 - \mathsf{a}_3 \geq 0, \mathsf{a}_2 - \mathsf{a}_4 \geq 0\}.$$

We notice that the inequality $\mathsf{a}_2 - \mathsf{a}_4 \geq 0$ is redundant because $(\mathsf{a}_2 - \mathsf{a}_4) = (\mathsf{a}_2 - \mathsf{a}_3) + (\mathsf{a}_3 - \mathsf{a}_4)$.

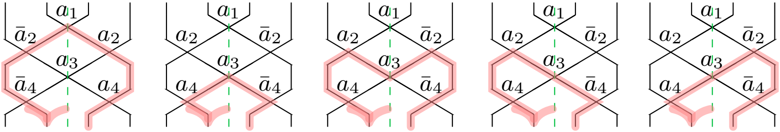
Symmetric rigorous paths

For a path $P = (\ell_{r_1} \to \cdots \to \ell_{r_{s+1}})$ in a symplectic wiring diagram $G^{\mathrm{symp}}(\boldsymbol{i}, k)$, its mirror $P^\vee$ is defined by

$$P^\vee := (\ell_{\overline{r_{s+1}}} \to \dots \ell_{\overline{r_1}}).$$

A path P with $k = n$ is called symmetric if $P = P^\vee$.

rigorous paths



symmetric	yes	yes	yes	no	no

Theorem [Cho–Fujita–L, in preparation]

The number of facets of the string cone $\mathcal{C}_{\boldsymbol{i}}^{(C_n)}$ is

$$\sum_{k=1}^{n-1} \#\{\text{rigorous paths in } G^{\mathrm{symp}}(\boldsymbol{i}, k)\} + \#\{\text{symmetric rigorous paths in } G^{\mathrm{symp}}(\boldsymbol{i}, n)\}.$$

Sketch of proof

1. We use the result [Fujita, 18]:

$$\Omega_{\boldsymbol{i}}^{A,C}(\mathcal{C}_{\boldsymbol{i}}^{(A_{2n-1})}) = \mathcal{C}_{\boldsymbol{i}}^{(C_n)}.$$

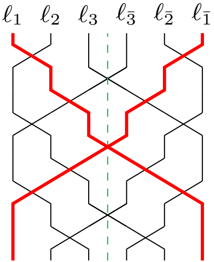
2. We use the description of the string cones in type A in terms of the wiring diagrams and rigorous paths in [Gleizer–Postnikov, 00].
3. To prove the non-redundancy of the string cone inequality obtained by a symmetric rigorous path in $G^{\mathrm{symp}}(\boldsymbol{i}, n)$, we use the non-redundancy of defining inequalities in type A.
4. To prove the redundancy of the string cone inequality obtained by a non-symmetric rigorous path in $G^{\mathrm{symp}}(\boldsymbol{i}, n)$, we construct appropriate *symmetric* rigorous paths which provides the given string cone inequality.

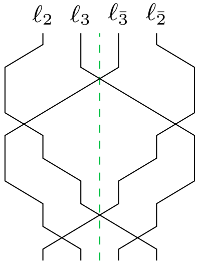
Contractions

cont($\boldsymbol{i}$): erase ℓ_1 and $\ell_{\bar{1}}$ and rearrangement.

Contraction maps a reduced word of the longest element in $W^{(C_n)}$ to a reduced word of the longest element in $W^{(C_{n-1})}$.

$\boldsymbol{i}$	cont($\boldsymbol{i}$)



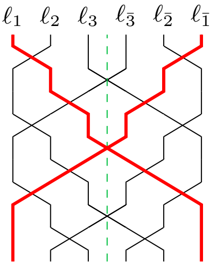
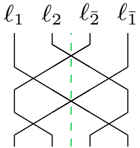


For $\boldsymbol{i} = (1,3,2,1,3,2,1,3,2)$, cont($\boldsymbol{i}$) $= (2,1,2,1)$.

Contractions

cont($\boldsymbol{i}$): erase ℓ_1 and $\ell_{\bar{1}}$ and rearrangement.

> Contraction maps a reduced word of the longest element in $W^{(C_n)}$ to a reduced word of the longest element in $W^{(C_{n-1})}$.

$\boldsymbol{i}$	cont($\boldsymbol{i}$)
	

For $\boldsymbol{i} = (1,3,2,1,3,2,1,3,2)$, cont($\boldsymbol{i}$) $= (2,1,2,1)$.

Contractions and the number of facets

Proposition [Cho–Fujita–L, in preparation]

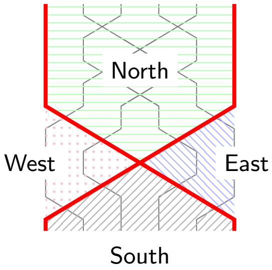
For $n > 2$, we have

$$\|\boldsymbol{i}\| \geq \|\mathrm{cont}(\boldsymbol{i})\| + (2n-1),$$

where $\|\boldsymbol{i}\|$ is the number of facets of $\mathcal{C}_{\boldsymbol{i}}^{(C)}$. Moreover, the equality hold if and only if $\boldsymbol{i}$ is the concatenation of

$$\mathrm{cont}(\boldsymbol{i}) \text{ and } (1, 2, \ldots, n, \ldots, 2, 1),$$

that is, there is no crossing except on the north sector of $G^{\mathrm{symp}}(\boldsymbol{i})$.



Simplicial string cones in type C

Theorem [Cho–Fujita–L, in preparation]

Let $\mathfrak{g}$ be a simple Lie algebra of type B_n or C_n with $n \geq 2$. Then, for a reduced word $\boldsymbol{i}$ of the longest element, the following are equivalent.

1. The number of facets of $\Delta_{\boldsymbol{i}}(\lambda)$ is $2N$ for every regular dominant integral weight λ.
2. The string cone $\mathcal{C}_{\boldsymbol{i}}$ is simplicial.
3. The reduced word $\boldsymbol{i}$ is either

$$\boldsymbol{i}_C = (n, n-1, n, n-1, n-2, n-1, n, n-1, n-2, \dots, 1, 2, \dots, n, \dots, 2, 1); \text{ or}$$
$$\boldsymbol{i}'_C := (n-1, n, n-1, n, n-2, n-1, n, n-1, n-2, \dots, 1, 2, \dots, n, \dots, 2, 1).$$

Gelfand–Testlin type string polytopes in type C

For $\boldsymbol{i}_C = (n, n-1, n, n-1, n-2, n-1, n, n-1, n-2, \dots, 1, 2, \dots, n, \dots, 2, 1)$ in type C,

$$\Delta_{\boldsymbol{i}_C}(\rho) \simeq \text{Gelfand–Tsetlin polytope } \mathrm{GT}_C(\rho)$$

by [Littelmann, 98].

Theorem [Cho–Fujita–L]

Let G be a simple Lie group of type C_n with $n \geq 2$ and $\boldsymbol{i}$ a reduced decomposition of the longest element. Then

$$\Delta_{\boldsymbol{i}}(\rho) \simeq \text{Gelfand–Tsetlin polytope } \mathrm{GT}_C(\rho)$$

if and only if $\boldsymbol{i} = \boldsymbol{i}_C$.

Key idea: Since the number of facets of $\mathrm{GT}_C(\rho)$ is $2N$, there are only two possibilities $\boldsymbol{i}_C$ and $\boldsymbol{i}'_C$. $\mathrm{GT}_C(\rho)$ is an integral polytope while $\Delta_{\boldsymbol{i}'_C}(\rho)$ is not integral.

Future work

- Studying other polytopes (i.e. toric degenerations) of G/B arising from cluster algebras.

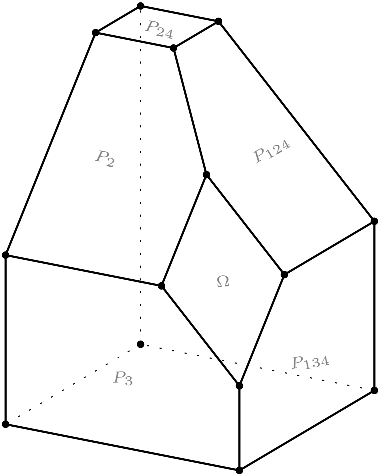
  There is an open embedding $U^-_{w_0} \hookrightarrow G/B$ and the the unipotent cell $U^-_{w_0}$ admits a cluster algebra structure. [Fujita–Oya, 20$^+$] constructed $\Delta(G/B, \mathcal{L}_\lambda, \nu_{\mathbf{s}})$ for each seed $\mathbf{s}$ and proved that

  $$\Delta(G/B, \mathcal{L}_\lambda, \nu_{\mathbf{s}}) \simeq \Delta_{\boldsymbol{i}}(\lambda) \text{ when } \mathbf{s} \text{ comes from } \boldsymbol{i}.$$

  In fact, the set of string polytopes is a (proper) subset of this larger family of Newton–Okounkov bodies.

Cartan–Killing type of G	A_2	A_3	A_4	B_2
cluster type	A_1	A_3	D_6	B_2
number of seeds	2	14	672	6
number of commutation classes*	2	8	62	2

*for $|i-j|$ with $c_{i,j} = 0$, we have $s_i s_j = s_j s_i$. This provides an equivalence relation on the reduced words. For example, $(1,3,2,1,3,2) \sim (3,1,2,1,3,2)$ in type A_3.



Question (work in progress)

Describe $\Delta(G/B, \mathcal{L}_\lambda, \nu_{\mathbf{s}})$ for various seeds $\mathbf{s}$ explicitly.

Thank you for your attention!